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An alternative determination of transverse shear stiffnesses for sandwich and laminated plates

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Abstract

Two-dimensional shear-deformable laminated plate theories can be classified as equivalent single-layer theories and layerwise theories. Layerwise theories lead to better approximations than equivalent single-layer theories, but the large number of independent unknowns in these theories requires more computational power in comparison with calculations, based on equivalent single-layer theories.

The quality of any equivalent single-layer theory based calculation is influenced by the correct determination of the effective stiffnesses. Many theories result in identical stiffnesses for bending, tension/compression, in-plane shear and torsion. The differences between the approaches are connected with the transverse shear stiffnesses. The method of determination of the transverse shear stiffnesses proposed here leads to expressions which depend on the solution of a Sturm–Liouville-problem. For special cases, the stiffnesses are calculated and compared with results from other authors. It can be shown that the present approach can be useful not only in the case of laminated plates, but also for sandwich plates. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: Sandwich; Laminate; Effective property; Transverse shear stiffness

1. Motivation

The classical plate theory based on Kirchhoff's hypotheses (Kirchhoff, 1850) shows a good agreement with experimental observations and three-dimensional solutions in the case of the global characteristics (e.g. deflections, eigenfrequencies), if the plates are made from metals and the behaviour can be described by geometrically linear equations. As composite materials (laminates, sandwiches) in some applications (e.g., aerospace industries) offer advantages over traditional materials, they increasingly substitute for these traditional materials. When using composites as a plate material, it is necessary to

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take into account that, e.g., the transverse shear modulus has a great influence on the structural mechanics analysis and should be determined accurately. The classical plate theory only predicts the response of thin isotropic plates with reasonable accuracy, yet it usually fails to yield similar accuracy for composite plates of similar configuration (Qi and Knight, 1996).

The reason for an incorrect determination of the global characteristics of plates made from sandwich or laminated materials by the classical plate theory is that Kirchhoff's theory ignores two transverse shear stresses. Starting with the pioneering works of E. Reissner (1944, 1945, 1947), Hencky (1947), Bollé (1947a, 1947b), Mindlin (1951) and others, there were several proposals for improved estimates of the global characteristics. Reissner introduced some assumptions for the stress distributions in the transverse direction, Hencky, Bollé and Mindlin, following Timoshenko's beam theory (Timoshenko, 1921), proposed an extended kinematical model with additional degrees of freedom for each material point of the reference (middle) surface. These ideas are the bases for the first refined plate theories for plates made from sandwich and laminated materials which are now called first order shear deformable theories (FOSDT).

The keypoint of the success of any FOSDT is the correct determination of the transverse shear stiffnesses. There are several proposals published (e.g. Bert, 1973; Whitney, 1973; Chow, 1971; Vlachoutsis, 1992; Qi and Knight, 1996; Knight and Qi, 1997) which are based on stress assumptions, kinematical assumptions or energy principles. A comprehensive analysis of different approaches was presented by (Rikards et al., 1990).

An alternative method of determination of the stiffnesses based on a theory of deformable directed surfaces and on the comparison of the eigenfrequencies of dual two-dimensional and three-dimensional problems was proposed for homogeneous plates and shells by Zhilin (1976). In the following, the determination of the effective stiffnesses is realized, comparing the forces and moments of dual problems in the case of plates whose material is inhomogeneous in the thickness direction. In the case of orthotropic materials, the expressions for all stiffnesses can be derived. Some numerical examples show the possibilities of the present approach, if the plate is made from a sandwich or a layered material with assumed isotropic properties.

2. Liznear basic plate equations

Let us consider the representation of a plate homogeneous or inhomogeneous in the thickness direction by a deformable surface. Each point of this surface is an infinitesimal small rigid body with 5 degrees of freedom (3 translations and 2 rotations). The kinematics for Cartesian coordinates are shown in Fig. 1 schematically. Using the continuum mechanics approach for formulating the basic linear plate equations and the consideration that our plate model is a two-dimensional elastic continuum, the following equations can be deduced in the case of small displacements, small rotations and under the assumption that the strain energy density is a quadratic form (Zhilin, 1976; Altenbach and Zhilin, 1988):

- the static equations of equilibrium

$$\nabla \cdot \mathbf{T} + \mathbf{q} = \mathbf{0},$$

$$\nabla \cdot \mathbf{M} + \mathbf{T}_{\times} + \mathbf{m} = \mathbf{0}. \tag{1}$$

$\mathbf{T}$, $\mathbf{M}$ are the force tensor (in-plane forces, transverse shear forces) and the couple tensor (bending couples, torsional couples), see (Fig. 2, $\mathbf{q}$, $\mathbf{m}$—the surface force and the surface couple vectors, $\mathbf{T}_{\times}$—the vector's invariant of the force tensor (e.g. Lurie, 1990), ∇—the nabla operator vector, $\cdot$—the scalar (inner) product and $(...)^{\mathrm{T}}$ denotes transposed.

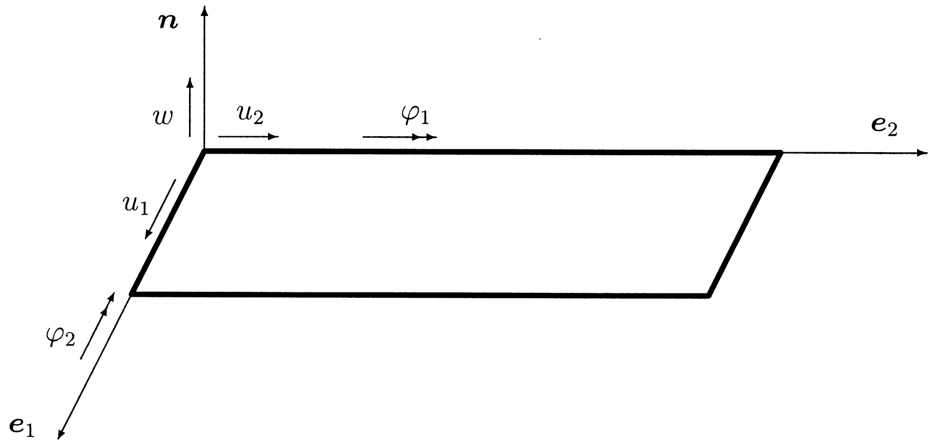

Fig. 1. Kinematics of the deformable directed surface.

- the linear geometric equations

$$\boldsymbol{\mu} = (\nabla \mathbf{u} \cdot \mathbf{a})^{\mathrm{sym}},$$

$$\boldsymbol{\gamma} = \nabla \mathbf{u} \cdot \mathbf{n} + \mathbf{c} \cdot \boldsymbol{\varphi},$$

$$\boldsymbol{\kappa} = \nabla \boldsymbol{\varphi}, \tag{2}$$

with **a**—the first metric tensor of the surface (e.g. Gould, 1988) and

$$\mathbf{c} = -\mathbf{a} \times \mathbf{n};$$

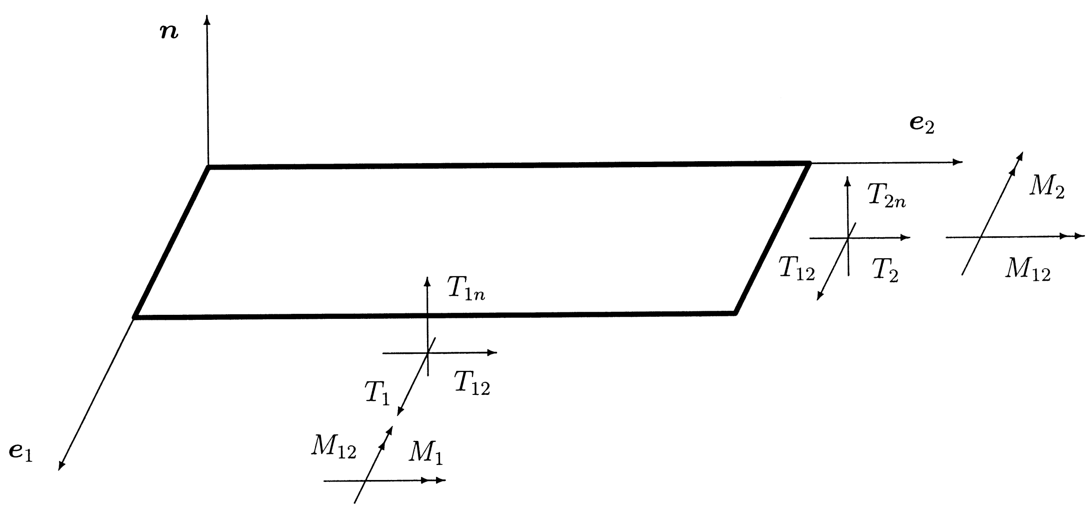

Fig. 2. Forces and couples of the deformable directed surface.

$\mathbf{n}$ denotes the normal unit vector to the surface, $\times$—vector product. $\boldsymbol{\mu}$, $\boldsymbol{\gamma}$ and $\boldsymbol{\kappa}$ are the in-plane deformation tensor, the transverse shear deformation vector and the flexural and torsional deformation tensor, respectively, and $\mathbf{u}$, $\boldsymbol{\varphi}$—the translation and the rotation vectors. $(...)^{\text{sym}}$ means the symmetric part.

- the strain energy density

$$W(\boldsymbol{\mu},\boldsymbol{\gamma},\boldsymbol{\kappa}) = \frac{1}{2}\boldsymbol{\mu}\cdot\cdot\mathbf{A}\cdot\cdot\boldsymbol{\mu} + \boldsymbol{\mu}\cdot\cdot\mathbf{B}\cdot\cdot\boldsymbol{\kappa} + \frac{1}{2}\boldsymbol{\kappa}\cdot\cdot\mathbf{C}\cdot\cdot\boldsymbol{\kappa} + \frac{1}{2}\boldsymbol{\gamma}\cdot\boldsymbol{\Gamma}\cdot\boldsymbol{\gamma} + \boldsymbol{\gamma}\cdot(\boldsymbol{\Gamma}_1\cdot\cdot\boldsymbol{\mu} + \boldsymbol{\Gamma}_2\cdot\cdot\boldsymbol{\kappa}) \tag{3}$$

Here, the 4th rank tensors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, the 3rd rank tensors $\boldsymbol{\gamma}_1$, $\boldsymbol{\gamma}_2$ and the 2nd rank tensor $\boldsymbol{\gamma}$ are stiffness tensors depending on the material properties and the thickness geometry.

- the constitutive equations

$$\mathbf{T}\cdot\mathbf{a} = \frac{\partial W}{\partial\boldsymbol{\mu}} = \mathbf{A}\cdot\cdot\boldsymbol{\mu} + \mathbf{B}\cdot\cdot\boldsymbol{\kappa} + \boldsymbol{\gamma}\cdot\boldsymbol{\Gamma}_1,$$

$$\mathbf{T}\cdot\mathbf{n} = \frac{\partial W}{\partial\boldsymbol{\gamma}} = \boldsymbol{\Gamma}\cdot\boldsymbol{\gamma} + \boldsymbol{\Gamma}_1\cdot\cdot\boldsymbol{\mu} + \boldsymbol{\Gamma}_2\cdot\cdot\boldsymbol{\kappa},$$

$$\mathbf{M}^T = \frac{\partial W}{\partial\boldsymbol{\kappa}} = \boldsymbol{\mu}\cdot\cdot\mathbf{B} + \mathbf{C}\cdot\cdot\boldsymbol{\kappa} + \boldsymbol{\gamma}\cdot\boldsymbol{\Gamma}_2 \tag{4}$$

The stiffness tensors contain a great number of components (36 in the general anisotropic case). For applications, there are two possibilities to reduce the number of linear independent components: the assumption of special cases of anisotropy and the assumption of symmetric properties in the thickness direction. Assuming a global orthotropic material behaviour (the Cartesian coordinate axes are the axes of orthotropy), the stiffness tensors have a special representation (Zhilin, 1976; Altenbach, 1987):

$$\mathbf{A} = A_{11}\mathbf{a}_1\mathbf{a}_1 + A_{12}(\mathbf{a}_1\mathbf{a}_2 + \mathbf{a}_2\mathbf{a}_1) + A_{22}\mathbf{a}_2\mathbf{a}_2 + A_{44}\mathbf{a}_4\mathbf{a}_4,$$

$$\mathbf{B} = B_{13}\mathbf{a}_1\mathbf{a}_3 + B_{14}\mathbf{a}_1\mathbf{a}_4 + B_{23}\mathbf{a}_2\mathbf{a}_3 + \mathbf{B}_{24}\mathbf{a}_2\mathbf{a}_4 + B_{42}\mathbf{a}_4\mathbf{a}_2,$$

$$\mathbf{C} = C_{22}\mathbf{a}_2\mathbf{a}_2 + C_{33}\mathbf{a}_3\mathbf{a}_3 + C_{34}(\mathbf{a}_3\mathbf{a}_4 + \mathbf{a}_4\mathbf{a}_3) + C_{44}\mathbf{a}_4\mathbf{a}_4,$$

$$\boldsymbol{\Gamma} = \Gamma_1\mathbf{a}_1 + \Gamma_2\mathbf{a}_2,$$

$$\boldsymbol{\Gamma}_1 = \mathbf{0}, \quad \boldsymbol{\Gamma}_2 = \mathbf{0}, \tag{5}$$

with

$$\mathbf{a}_1 = \mathbf{a} = \mathbf{e}_1\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_2,$$

$$\mathbf{a}_2 = \mathbf{e}_1\mathbf{e}_1 - \mathbf{e}_2\mathbf{e}_2,$$

$$\mathbf{a}_3 = \mathbf{c} = \mathbf{e}_1\mathbf{e}_2 - \mathbf{e}_2\mathbf{e}_1,$$

$$\mathbf{a}_4 = \mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1;$$

$\mathbf{e}_1$, $\mathbf{e}_2$ are the unit basic vectors for the surface coordinates. The unit vectors are orthogonal, so that the tensors $\mathbf{a}_i$ (i = 1,2,3,4) are connected by

$$\frac{1}{2}\mathbf{a}_i \cdot\cdot\, \mathbf{a}_j = \delta_{ij},$$

$$\delta_{ij} = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$$

For the use of Eq. (5), the problem of identification of the stiffnesses, A_{11}, A_{22}, A_{12}, A_{44}, B_{13}, B_{14}, B_{23}, B_{24}, B_{42}, C_{33}, C_{22}, C_{34}, C_{44}, Γ_1 and Γ_2, should be solved. The identification procedures are based on the solution of static or dynamic problems with the help of the two-dimensional theory (assuming a deformable surface) and the three-dimensional theory (taking into account the third direction—the thickness). For the determination of the stiffnesses we have to compare both solutions for the averaged stresses or the eigenfrequencies. In the case of sandwich or laminated plates, we prefer static problems. A suitable method of identification will be presented in the next section. The use of dynamical characteristics for the identification of stiffnesses in the case of homogeneous plates is presented by Zhilin (1976).

Until now, no answer has been given to the question what kind of static problems we have to use for the identification. So the best choice are problems leading to simple solutions in the two-dimensional and the three-dimensional cases. Below, we discuss problems related to rectangular deformable directed surfaces and three-dimensional thin bodies with a rectangular middle surface. This assumption leads to the following representations (x_1, x_2, z Cartesian coordinates, 1, 2, n indices connected with the Cartesian coordinates) of:

- the displacement and the rotation vectors

$$\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + w\mathbf{n}$$

$$\boldsymbol{\varphi} = -\varphi_2\mathbf{e}_1 + \varphi_1\mathbf{e}_2, \tag{6}$$

u_1, u_2 are the in-plane displacements, w—the deflections and φ_1, φ_2—the rotations (Fig. 1).

- the force and the couple tensors

$$\mathbf{T} = T_1\mathbf{e}_1\mathbf{e}_1 + T_{12}(\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1) + T_2\mathbf{e}_2\mathbf{e}_2 + T_{1n}\mathbf{e}_1\mathbf{n} + T_{2n}\mathbf{e}_2\mathbf{n},$$

$$\mathbf{M} = M_1\mathbf{e}_1\mathbf{e}_2 - M_{12}(\mathbf{e}_1\mathbf{e}_1 - \mathbf{e}_2\mathbf{e}_2) - M_2\mathbf{e}_2\mathbf{e}_1. \tag{7}$$

T_1, T_{12} and T_2 are the in-plane forces; T_{1n} and T_{2n}—the out-of-plane forces; M_1 and M_2—the bending couples and M_{12}—the torsion couples (Fig. 2).

- the deformation tensors

$$\boldsymbol{\mu} = \mu_1\mathbf{e}_1\mathbf{e}_1 + \mu_{12}(\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1) + \mu_2\mathbf{e}_2\mathbf{e}_2,$$

$$\boldsymbol{\gamma} = \gamma_1\mathbf{e}_1 + \gamma_2\mathbf{e}_2,$$

$$\boldsymbol{\kappa} = \kappa_1\mathbf{e}_1\mathbf{e}_2 - \kappa_{12}\mathbf{e}_1\mathbf{e}_1 + \kappa_{21}\mathbf{e}_2\mathbf{e}_2 - \kappa_2\mathbf{e}_2\mathbf{e}_1. \tag{8}$$

3. Identification of the stiffnesses

3.1. Orthotropic material behaviour

The starting point of the identification of all stiffnesses is the generalized Hooke's law with respect to orthotropic material behaviour (the axes of orthotropy are parallel to the coordinate axes) (e.g. Lai et al., 1993)

$$\varepsilon_1 = u^*_{1,1} = \frac{1}{E_1}\sigma_1 - \frac{\nu_{21}}{E_2}\sigma_2 - \frac{\nu_{n1}}{E_n}\sigma_n,$$

$$\varepsilon_2 = u^*_{2,2} = \frac{1}{E_2}\sigma_2 - \frac{\nu_{12}}{E_1}\sigma_1 - \frac{\nu_{n2}}{E_n}\sigma_n,$$

$$\varepsilon_n = u^*_{n,n} = \frac{1}{E_n}\sigma_n - \frac{\nu_{1n}}{E_1}\sigma_1 - \frac{\nu_{2n}}{E_2}\sigma_2,$$

$$\gamma_{12} = u^*_{1,2} + u^*_{2,1} = \frac{\tau_{12}}{G_{12}},$$

$$\gamma_{n1} = u^*_{1,n} + u^*_{n,1} = \frac{\tau_{n1}}{G_{n1}},$$

$$\gamma_{2n} = u^*_{2,n} + u^*_{n,2} = \frac{\tau_{2n}}{G_{2n}}, \tag{9}$$

with $\mathbf{u}^*$ as the displacement vector when the three-dimensional elasticity is used and

$$\nu_{ij}E_j = \nu_{ji}E_i.$$

In the case of plates that are inhomogeneous in the thickness direction, all material parameters (Young's moduli: E_i in the i-direction; Poisson's ratio: ν_{ij} for the transverse strain in the j-direction when stressed in the i-th direction; and shear moduli: G_{ij} in the ij-plane) are functions of the coordinate in the thickness direction, e.g. $E_i = E_i(z)$.

Below, we discuss the solutions of three static problems from the viewpoint of the two-dimensional and the three-dimensional theories. From the first two problems, we get the classical stretching, in-plane shear, bending, torsion and coupling stiffnesses. From the last problem, the transverse shear stiffnesses can be derived. The comparison of both two-dimensional and three-dimensional solutions is based on calculated forces and couples (two-dimensional solutions) and averaged stresses (three-dimensional case). The averaging can be done in the case of plates by very simple formulae:

$$\mathbf{T} = \langle \mathbf{a} \cdot \boldsymbol{\sigma} \rangle, \quad \mathbf{M} = \langle \mathbf{a} \cdot \boldsymbol{\sigma} z \cdot \mathbf{c} \rangle, \tag{10}$$

$\boldsymbol{\sigma}$ denotes the stress tensor and $\langle(...)\rangle$ the integration over the plate thickness

$$\langle(\ldots)\rangle = \int_{-h/2}^{h/2} (\ldots)\mathrm{d}z.$$

3.2. Classical stiffnesses

3.2.1. Problem 1: bending and stretching

From the following kinematical assumption, we get the two-dimensional solution

$$\mathbf{u} = D_1 x_1 \mathbf{e}_1 + D_2 x_2 \mathbf{e}_2 - \frac{1}{2}\left(\frac{x_1^2}{R_1} + \frac{x_2^2}{R_2}\right)\mathbf{n}, \quad \boldsymbol{\varphi} = -\frac{x_2}{R_2}\mathbf{e}_1 + \frac{x_1}{R_1}\mathbf{e}_2, \tag{11}$$

D_1, D_2, R_1 and R_2 are constants. The kinematical field Eq. (11) corresponds to Kirchhoff's plate behaviour (Kirchhoff, 1850). From Eq. (2), we calculate

$$\boldsymbol{\mu} = D_1 \mathbf{e}_1\mathbf{e}_1 + D_2 \mathbf{e}_2\mathbf{e}_2, \quad \boldsymbol{\gamma} = 0, \quad \boldsymbol{\kappa} = \frac{1}{R_1}\mathbf{e}_1\mathbf{e}_2 - \frac{1}{R_2}\mathbf{e}_2\mathbf{e}_1 \tag{12}$$

With respect to the constitutive equations Eq. (4), we finally obtain:

$$T_1 = D_1(A_{11} + 2A_{12} + A_{22}) + D_2(A_{11} - A_{22}) - \frac{B_{13} + B_{23} - B_{14} - B_{24}}{R_1} - \frac{B_{13} + B_{23} + B_{14} + B_{24}}{R_2},$$

$$T_2 = D_1(A_{11} - A_{22}) + D_2(A_{11} - 2A_{12} + A_{22}) - \frac{B_{13} - B_{23} - B_{14} + B_{24}}{R_1} - \frac{B_{13} - B_{23} + B_{14} - B_{24}}{R_2},$$

$$M_1 = D_1(-B_{13} - B_{23} + B_{14} + B_{24}) - D_2(B_{13} - B_{23} - B_{14} + B_{24}) + \frac{C_{33} - 2C_{34} + C_{44}}{R_1} + \frac{C_{33} - C_{44}}{R_2},$$

$$M_2 = -D_1(B_{13} + B_{23} + B_{14} + B_{24}) - D_2(B_{13} - B_{23} + B_{14} - B_{24}) + \frac{C_{33} - C_{44}}{R_1} + \frac{C_{33} + 2C_{34} + C_{44}}{R_2}. \tag{13}$$

The nonzero three-dimensional strain tensor components that are dual to the two-dimensional deformation tensors Eq. (12) are

$$\varepsilon_1 = D_1 + \frac{z}{R_1},$$

$$\varepsilon_2 = D_2 + \frac{z}{R_2}. \tag{14}$$

Assuming the state of plane stress from Hooke's law Eq. (9), we can calculate the stresses:

$$\sigma_1 = \frac{E_1}{1 - \nu_{12}\nu_{21}}(\varepsilon_1 + \nu_{21}\varepsilon_2),$$

$$\sigma_2 = \frac{E_2}{1 - \nu_{12}\nu_{21}}(\varepsilon_2 + \nu_{12}\varepsilon_1). \tag{15}$$

Taking Eq. (14) into account, the integration of the stresses over the thickness results in:

$$T_1 = D_1\left\langle\frac{E_1}{1 - \nu_{12}\nu_{21}}\right\rangle + \frac{1}{R_1}\left\langle\frac{E_1 z}{1 - \nu_{12}\nu_{21}}\right\rangle + D_2\left\langle\frac{\nu_{21}E_1}{1 - c\nu_{12}\nu_{21}}\right\rangle + \frac{1}{R_2}\left\langle\frac{\nu_{21}E_1 z}{1 - \nu_{12}\nu_{21}}\right\rangle,$$

$$T_2 = D_2\left\langle\frac{E_2}{1-\nu_{12}\nu_{21}}\right\rangle + \frac{1}{R_2}\left\langle\frac{E_2 z}{1-\nu_{12}\nu_{21}}\right\rangle + D_1\left\langle\frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}}\right\rangle + \frac{1}{R_1}\left\langle\frac{\nu_{12}E_2 z}{1-\nu_{12}\nu_{21}}\right\rangle,$$

$$M_1 = D_1\left\langle\frac{E_1 z}{1-\nu_{12}\nu_{21}}\right\rangle + \frac{1}{R_1}\left\langle\frac{E_1 z^2}{1-\nu_{12}\nu_{21}}\right\rangle + D_2\left\langle\frac{\nu_{21}E_1 z}{1-\nu_{12}\nu_{21}}\right\rangle + \frac{1}{R_2}\left\langle\frac{\nu_{21}E_1 z^2}{1-\nu_{12}\nu_{21}}\right\rangle,$$

$$M_2 = D_2\left\langle\frac{E_2 z}{1-\nu_{12}\nu_{21}}\right\rangle + \frac{1}{R_2}\left\langle\frac{E_2 z^2}{1-\nu_{12}\nu_{21}}\right\rangle + D_1\left\langle\frac{\nu_{12}E_2 z}{1-\nu_{12}\nu_{21}}\right\rangle + \frac{1}{R_1}\left\langle\frac{\nu_{12}E_2 z^2}{1-\nu_{12}\nu_{21}}\right\rangle. \tag{16}$$

The comparison of Eqs. (13) and (16) leads to ten unknown effective stiffnesses:

$$A_{11} = \frac{1}{4}\left\langle\frac{E_1+E_2+2E_1\nu_{21}}{1-\nu_{12}\nu_{21}}\right\rangle,$$

$$A_{12} = \frac{1}{4}\left\langle\frac{E_1-E_2}{1-\nu_{12}\nu_{21}}\right\rangle,$$

$$A_{22} = \frac{1}{4}\left\langle\frac{E_1+E_2-2E_1\nu_{21}}{1-\nu_{12}\nu_{21}}\right\rangle,$$

$$B_{13} = -\frac{1}{4}\left\langle\frac{E_1+E_2+2E_1\nu_{21}}{1-\nu_{12}\nu_{21}}z\right\rangle,$$

$$B_{23} = -B_{14} = -\frac{1}{4}\left\langle\frac{E_1-E_2}{1-\nu_{12}\nu_{21}}z\right\rangle,$$

$$B_{24} = \frac{1}{4}\left\langle\frac{E_1+E_2-2E_1\nu_{21}}{1-\nu_{12}\nu_{21}}z\right\rangle,$$

$$C_{33} = \frac{1}{4}\left\langle\frac{E_1+E_2+2E_1\nu_{21}}{1-\nu_{12}\nu_{21}}z^2\right\rangle,$$

$$C_{34} = -\frac{1}{4}\left\langle\frac{E_1-E_2}{1-\nu_{12}\nu_{21}}z^2\right\rangle,$$

$$C_{44} = \frac{1}{4}\left\langle\frac{E_1+E_2-2E_1\nu_{21}}{1-\nu_{12}\nu_{21}}z^2\right\rangle \tag{17}$$

3.2.2. Problem 2: in-plane shear

Now we start from the following kinematical assumptions for the two-dimensional deformable surface:

$$\mathbf{u} = S_1 x_2 \mathbf{e}_1 + S_1 x_1 \mathbf{e}_2 - S_2 x_1 x_2 \mathbf{n}, \quad \boldsymbol{\varphi} = -S_2(x_1 \mathbf{e}_1 - x_2 \mathbf{e}_2) \tag{18}$$

The dual three-dimensional strain tensor components are:

$$\gamma_{12} = u^*_{1,2} + u^*_{2,1} = S_1 + S_2 z \tag{19}$$

Repeating the calculations of Problem 1 we finally get:

$$A_{44} = \langle G_{12} \rangle,$$

$$B_{42} = -\langle G_{12} z \rangle,$$

$$C_{22} = \left\langle G_{12} z^2 \right\rangle \tag{20}$$

3.3. Transverse shear stiffnesses

3.3.1. Problem 3: torsion

Considering a two-dimensional deformable surface ($|x_1| \leq l_1$, $|x_2| < \infty$) with constant torsional couples on boundaries $x_1 = \pm l_1$, we assume the following displacements and rotations:

$$\mathbf{u} = u_2(x_1)\mathbf{e}_2,$$

$$\boldsymbol{\varphi} = -\varphi_2(x_1)\mathbf{e}_1. \tag{21}$$

After the determination of the deformations tensors by Eq. (2), we can calculate the force and couple tensors from the constitutive equations Eq. (4):

$$\mathbf{T} = (A_{44} u_{2,1} - B_{42} \varphi_{2,1})\mathbf{a}_4 + (\Gamma_1 - \Gamma_2)\varphi_2 \mathbf{e}_2 \mathbf{n},$$

$$\mathbf{M} = (B_{42} u_{2,1} - C_{22} \varphi_{2,1})\mathbf{a}_2. \tag{22}$$

Both tensors should satisfy the equations of equilibrium Eq. (1), so we get

$$A_{44} u_{2,11} - B_{42} \varphi_{2,11} = 0,$$

$$B_{42} u_{2,11} - C_{22} \varphi_{2,11} + (\Gamma_1 - \Gamma_2)\varphi_2 = 0. \tag{23}$$

Taking into account the boundary conditions on $x_1 = \pm l_1$:

$$\pm \mathbf{e}_1 \cdot \mathbf{T} = \mathbf{0},$$

$$\pm \mathbf{e}_1 \cdot \mathbf{M} = \pm M^*_{12} \mathbf{e}_1$$

(M^*_{12} is a constant torsional couple), the solutions of Eq. (23) are:

$$u_2 = -M^*_{12} \frac{B_{42}}{A_{44} C_{22} - B_{42}^2} \frac{\sinh \lambda x_1}{\lambda \cosh \lambda l_1},$$

$$\varphi_2 = -M^*_{12} \frac{A_{44}}{A_{44} C_{22} - B_{42}^2} \frac{\sinh \lambda x_1}{\lambda \cosh \lambda l_1}, \tag{24}$$

with

$$\lambda^2 = \frac{(\Gamma_1 - \Gamma_2)A_{44}}{\left(A_{44}C_{22} - B_{42}^2\right)}.$$

From Eq. (24), we calculate the force T_{12} and the couple M_{12}

$$T_{12} = 0, \quad M_{12} = M_{12}^* \frac{\cosh \lambda x_1}{\cosh \lambda l_1} \tag{25}$$

The dual solution for a three-dimensional elastic strip ($|x_1| \leq l_1$, $|x_2| < \infty$, $|z| \leq h/2$) can be derived from the following displacement field: $u_1^* = w^* = 0$, $u_2^* = u_2^*(x_1,z)$. The corresponding stress tensor is

$$\boldsymbol{\sigma} = G_{12}\frac{\partial u_2^*}{\partial x_1}\mathbf{a}_4 + G_{2n}\frac{\partial u_2^*}{\partial z}(\mathbf{e}_2\mathbf{n} + \mathbf{n}\mathbf{e}_2). \tag{26}$$

From the three-dimensional equilibrium equations (e.g. Lai et al., 1993), it follows that

$$G_{12}\frac{\partial^2 u_2^*}{\partial x_1^2} + \frac{\partial}{\partial z}\left(G_{2n}\frac{\partial u_2^*}{\partial z}\right) = 0. \tag{27}$$

The solution can be derived under the assumption $\sigma_n = \tau_{1n} = \tau_{2n} = 0$ at $|z| = h/2$. Using Fourier's separation method, $u_2^*(x_1,z) = X(x_1)Z(z)$, we finally get a Sturm–Liouville problem:

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(G_{2n}\frac{\mathrm{d}Z}{\mathrm{d}z}\right) + \lambda_*^2 G_{12} Z = 0, \quad \left.\frac{\mathrm{d}Z}{\mathrm{d}z}\right|_{|z|=\frac{h}{2}} = 0, \tag{28}$$

and

$$\frac{\mathrm{d}^2 X}{\mathrm{d}x_1^2} - \lambda_*^2 X = 0. \tag{29}$$

From all solutions of Eq. (28), we select the lowest nontrivial positive value λ_*. The solution of Eq. (29) then results in:

$$X(x_1) = B\frac{\sinh \lambda_* x_1}{\lambda_* \cosh \lambda_* l_1}$$

and the displacement u_2^* can be expressed by

$$u_2^* = BZ(z)\frac{\sinh \lambda_* x_1}{\lambda_* \cosh \lambda_* l_1}.$$

From this displacement field, the force T_{12} and the couple M_{12} follow:

$$T_{12} = \langle \tau_{12} \rangle = B\big\langle G_{12} Z(z)\big\rangle \frac{\cosh \lambda_* x_1}{\cosh \lambda_* l_1},$$

$$M_{12} = -\langle \tau_{12} z \rangle = -B\big\langle G_{12} Z(z) z\big\rangle \frac{\cosh \lambda_* x_1}{\cosh \lambda_* l_1}.$$

Integrating Eq. (28) and taking into account the boundary conditions for $Z(z)$, the identity

$\langle G_{12}Z(z)\rangle = 0$ is deduced. The constant B can be computed from the condition that at $x_1 = l_1$ $M_{12} = M_{12}^*$. After some calculation, we find

$$T_{12} = 0, \quad M_{12} = M_{12}^* \frac{\cosh \lambda_* x_1}{\cosh \lambda_* l_1}, \quad u_2^* = -\frac{Z(z) M_{12}^*}{\langle G_{12} Z(z) z\rangle} \frac{\sinh \lambda_* x_1}{\lambda_* \cosh \lambda_* l_1}.$$

From the comparison of both the two-dimensional and the three-dimensional solution, we conclude

$$\lambda = \lambda_* = \sqrt{\frac{(\Gamma_1 - \Gamma_2)A_{44}}{A_{44}C_{22} - B_{42}^2}}. \tag{30}$$

In this case, we obtain the same values for the force T_{12} and the couple M_{12}, calculated by the two-dimensional and the three-dimensional theories. If we compare both kinematic fields, we must demand that, in the weighted least square sense,

$$\left\langle G_{12}\left(u_2^* - u_2 - z\varphi_2\right)^2\right\rangle = \min_{u_2, \varphi_2}.$$

The stationary conditions result in

$$u_2 = \frac{M_{12}^* \langle G_{12} z\rangle}{\langle G_{12}\rangle\langle G_{12} z^2\rangle - \langle G_{12} z^2\rangle} \frac{\sinh \lambda_* x_1}{\lambda_* \cosh \lambda_* l_1},$$

$$\varphi_2 = -\frac{M_{12}^* \langle G_{12}\rangle}{\langle G_{12}\rangle\langle G_{12} z^2\rangle - \langle G_{12} z^2\rangle} \frac{\sinh \lambda_* x_1}{\lambda_* \cosh \lambda_* l_1}.$$

This solution is equivalent to Eq. (24).

So we find only one equation Eq. (30) for the determination of the two stiffnesses Γ_1, Γ_2. From the corresponding problems for a two-dimensional strip ($|x_1| < \infty$, $|x_2| \leq l_2$) and the dual three-dimensional strip ($|x_1| < \infty$, $|x_2| \leq l_2$, $|z| \leq h/2$) under constant torsional moments at $|x_2| \leq l_2$, we get

$$\eta = \sqrt{\frac{(\Gamma_1 + \Gamma_2)A_{44}}{A_{44}C_{22} - B_{42}^2}}, \tag{31}$$

η is the lowest nontrivial positive solution of the following Sturm–Liouville problem:

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(G_{1n}\frac{\mathrm{d}\tilde{Z}}{\mathrm{d}z}\right) + \eta^2 G_{12}\tilde{Z} = 0, \quad \left.\frac{\mathrm{d}\tilde{Z}}{\mathrm{d}z}\right|_{|z| = \frac{h}{2}} = 0.$$

Finally, from Eqs. (30) and (31), we can calculate the stiffnesses:

$$\Gamma_1 = \frac{1}{2}(\lambda^2 + \eta^2)\frac{A_{44}C_{22} - B_{42}^2}{A_{44}},$$

$$\Gamma_2 = \frac{1}{2}(\eta^2 - \lambda^2)\frac{A_{44}C_{22} - B_{42}^2}{A_{44}}. \tag{32}$$

Remark: We have obtained all 15 stiffnesses which take place in the theory of deformable directed surfaces. Most of them Eq. (17) are the same as in the standard textbooks in the theory of laminated plates (e.g. Altenbach et al., 1996). These stiffnesses are some average values of the material properties and the thickness geometry over the thickness. The last two stiffnesses (Γ_1, Γ_2) are different from the well-known proposals because they depend on the solution of two Sturm–Liouville problems. The following discussion of special cases shows that these non-traditional stiffnesses are useful in a wide range of cross-section configurations. They can be recommended for both laminates and sandwiches.

4. Examples

4.1. Isotropic homogeneous plate

Assuming a plate made from isotropic material with homogeneous properties in the thickness direction, the stiffnesses can be calculated by formulae following from the general expressions (Eqs. (17), (20) and (32)). The symmetry of the material properties and the geometrical parameters cause the vanishing values of the coupling stiffnesses, that means $\mathbf{B} \equiv \mathbf{0}$. The membrane and the plate stiffnesses can be estimated by:

$$A_{11} = \frac{1}{2}\frac{Eh}{1-\nu},$$

$$A_{12} = 0,$$

$$A_{22} = \frac{1}{2}\frac{Eh}{1+\nu},$$

$$A_{44} = Gh = A_{22},$$

$$C_{33} = \frac{Eh^3}{24(1-\nu)},$$

$$C_{34} = 0,$$

$$C_{44} = \frac{Eh^3}{24(1+\nu)},$$

$$C_{22} = \frac{Gh^3}{12} = C_{44}. \tag{33}$$

E, ν, $G = E/2(1+\nu)$ are the Young's modulus, the Poisson's ratio and the shear modulus, respectively, of the isotropic material which are constant with respect to the thickness coordinate z. The classical plate stiffness (flexural rigidity) results from

$$C_{33} + C_{44} = \frac{Eh^3}{12(1-\nu^2)}.$$

This is the same value as in the classical textbooks (e.g. Love, 1944).

The transverse stiffness results from Eq. (30) with $B_{42} = 0$,

$$\Gamma = \lambda^2 C_{22},$$

where λ is the solution of the following Sturm–Liouville problem:

$$\frac{\mathrm{d}^2 Z}{\mathrm{d}z^2} + \lambda^2 Z = 0,$$

$$\left.\frac{\mathrm{d}Z}{\mathrm{d}z}\right|_{|z|=\frac{h}{2}} = 0.$$

Its solution, $\cos \lambda z = 0$, leads to the lowest nontrivial positive eigenvalue, $\lambda = \pi/h$. We finally get

$$\Gamma = \frac{\pi^2}{h^2}\frac{Gh^3}{12} = \frac{\pi^2}{12}Gh.$$

The transverse shear stiffness is determined with a shear correction coefficient proposed first by Mindlin (1951). This value is very close to Reissner's proposal (instead of $\pi^2/12$ in Reissner (1944), the shear correction was 5/6).

4.2. Sandwich plate

Let us consider a typical sandwich (Fig. 3) with a very soft core. We assume that $h_c \gg h_f$ and $E_c \ll E_f$, $G_c \ll G_f$. Each layer is isotropic, that means $G_c = E_c/2(1+\nu_c)$ etc. In addition, we presume the symmetry in the thickness direction ($\mathbf{B} \equiv \mathbf{0}$). From Eqs. (17) and (20), we can calculate ($h = h_c + h_f$):

$$A_{11} = \frac{1}{2}\left(\frac{E_f h_f}{1-\nu_f} + \frac{E_c h_c}{1-\nu_c}\right),$$

$$A_{22} = \frac{1}{2}\left(\frac{E_f h_f}{1+\nu_f} + \frac{E_c h_c}{1+\nu_c}\right),$$

$$A_{44} = G_f h_f + G_c h_c,$$

$$C_{33} = \frac{1}{24}\left[\frac{E_f\left(h^3 - h_c^3\right)}{1-\nu_f} + \frac{E_c h_c^3}{1-\nu_c}\right],$$

$$C_{44} = \frac{1}{24}\left[\frac{E_f\left(h^3 - h_c^3\right)}{1+\nu_f} + \frac{E_c h_c}{1+\nu_c}\right],$$

$$C_{22} = \frac{1}{12}\left[G_f\left(h^3 - h_c^3\right) + G_c h_c^3\right] = C_{44}. \tag{34}$$

The flexural rigidity of the sandwich follows from

$$C_{33} + C_{44} = \frac{1}{12}\left[\frac{E_f\left(h^3 - h_c^3\right)}{1 - \nu_f^2} + \frac{E_c h_c^3}{1 - \nu_c^2}\right].$$

The transverse shear stiffness in the case of sandwiches follows from the same formula as in Section 4.1,

$$\Gamma = \lambda^2 C_{22}, \tag{35}$$

but the determination of λ is different. Now we assume the following stress tensor:

$$\boldsymbol{\sigma} = \begin{cases} G_f\left[\dfrac{\partial u_2^*}{\partial x_1}\mathbf{a}_4 + \dfrac{\partial u_2^*}{\partial z}(\mathbf{e}_2\mathbf{n} + \mathbf{n}\mathbf{e}_2)\right] & \dfrac{h_c}{2} \le |z| \le \dfrac{h}{2}, \\ G_c\left[\dfrac{\partial u_2^*}{\partial x_1}\mathbf{a}_4 + \dfrac{\partial u_2^*}{\partial z}(\mathbf{e}_2\mathbf{n} + \mathbf{n}\mathbf{e}_2)\right] & |z| \le \dfrac{h_c}{2} \end{cases}.$$

The boundary conditions on the upper and the lower boundaries ($|z| = h/2$) are stress-free conditions

$$\left.\frac{\partial u_2^*}{\partial z}\right|_{|z|=\frac{h}{2}} = 0.$$

In addition, we have to fulfil the continuity conditions on $|z| = h_c/2$,

$$u_2^*\Big|_{|z|=\frac{h_c}{2}-0} = u_2^*\Big|_{|z|=\frac{h_c}{2}+0},$$

$$G_c\left.\frac{\partial u_2^*}{\partial z}\right|_{|z|=\frac{h_c}{2}-0} = G_f\left.\frac{\partial u_2^*}{\partial z}\right|_{|z|=\frac{h_c}{2}+0}.$$

Using Fourier's separation method, $u_2^*(x_1,z) = X(x_1)Z(z)$, we get the following eigenvalue problem:

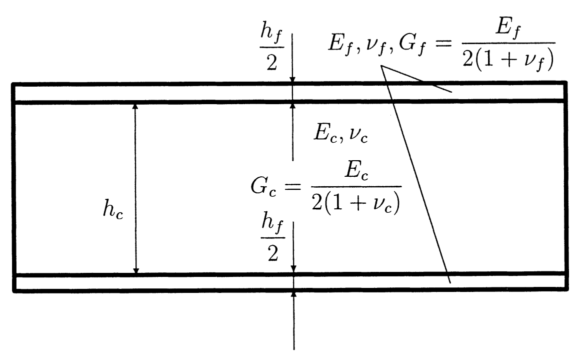

Fig. 3. Geometrical and material properties of a typical sandwich.

$$G_f\left(\frac{\partial^2 Z}{\partial z^2}+\lambda^2 Z\right)=0\ \frac{h_c}{2}\le|z|\le\frac{h}{2},$$

$$G_c\left(\frac{\partial^2 Z}{\partial z^2}+\lambda^2 Z\right)=0\ |z|\le\frac{h_c}{2},$$

with

$$\left.\frac{\partial Z}{\partial z}\right|_{|z|=\frac{h}{2}}=0,\quad Z|_{|z|=\frac{h_c}{2}-0}=Z|_{|z|=\frac{h_c}{2}+0},$$

$$G_c\left.\frac{\partial Z}{\partial z}\right|_{|z|=\frac{h_c}{2}-0}=G_f\left.\frac{\partial Z}{\partial z}\right|_{|z|=\frac{h_c}{2}+0}$$

and $Z(z)=-Z(-z)$. The solution can be obtained by

$$Z(z)=\begin{cases}A\cos\lambda\left(z-\frac{h}{2}\right) & \frac{h_c}{2}\le z\le\frac{h}{2},\\ B\sin\lambda z & |z|\le\frac{h_c}{2},\\ -A\cos\lambda\left(z+\frac{h}{2}\right) & -\frac{h}{2}\le z\le-\frac{h_c}{2}\end{cases}.$$

Replacing G_c/G_f by μ, the constants A and B should satisfy the system

$$A\cos\lambda\frac{h_f}{2}-B\sin\lambda\frac{h_c}{2}=0,$$

$$A\sin\lambda\frac{h_f}{2}-B\mu\cos\lambda\frac{h_c}{2}=0.$$

This system of equations has a solution if the determinant is zero

$$\mu\cos\lambda\frac{h_f}{2}\cos\lambda\frac{h_c}{2}-\sin\lambda\frac{h_f}{2}\sin\lambda\frac{h_c}{2}=0.$$

Introducing $\gamma=\lambda h/2$ and $\alpha=h_c/h$, finally we get

$$\mu\cos\gamma(1-\alpha)\cos\gamma\alpha-\sin\gamma(1-\alpha)\sin\gamma\alpha=0, \tag{36}$$

with $0\le\mu<\infty$ and $0\le\alpha\le1$. In the case of very soft cores, the plate stiffness (flexural rigidity) and the transverse shear stiffness can be computed approximately

$$C_{33}+C_{44}=\frac{1}{4}\frac{E_f h^2 h_f}{1-\nu_f^2},$$

$$\Gamma = G_c h, \tag{37}$$

whenever μ is smaller than the ratio h_f/h. Both values Eq. (37) were first published by Reissner (1947).

4.3. Three-layer plate

The effective stiffnesses of laminated plates can be calculated by several approaches corresponding to the mechanics of composites (e.g. Christensen, 1979). There are two approaches which can be regarded as boundaries (Hill, 1964): the isostrain or acting in parallel model (Voigt) and the isostress or acting in series model (Reuss) (e.g. Chawla, 1987). It is well known that the membrane and the plate stiffnesses can be identified by the assumption of the acting in parallel model. For the transverse shear stiffnesses, this assumption can result in some inaccuracies in dependence on the thicknesses and the material properties of each layer. As an example, we consider a three-layer system similar to the sandwich

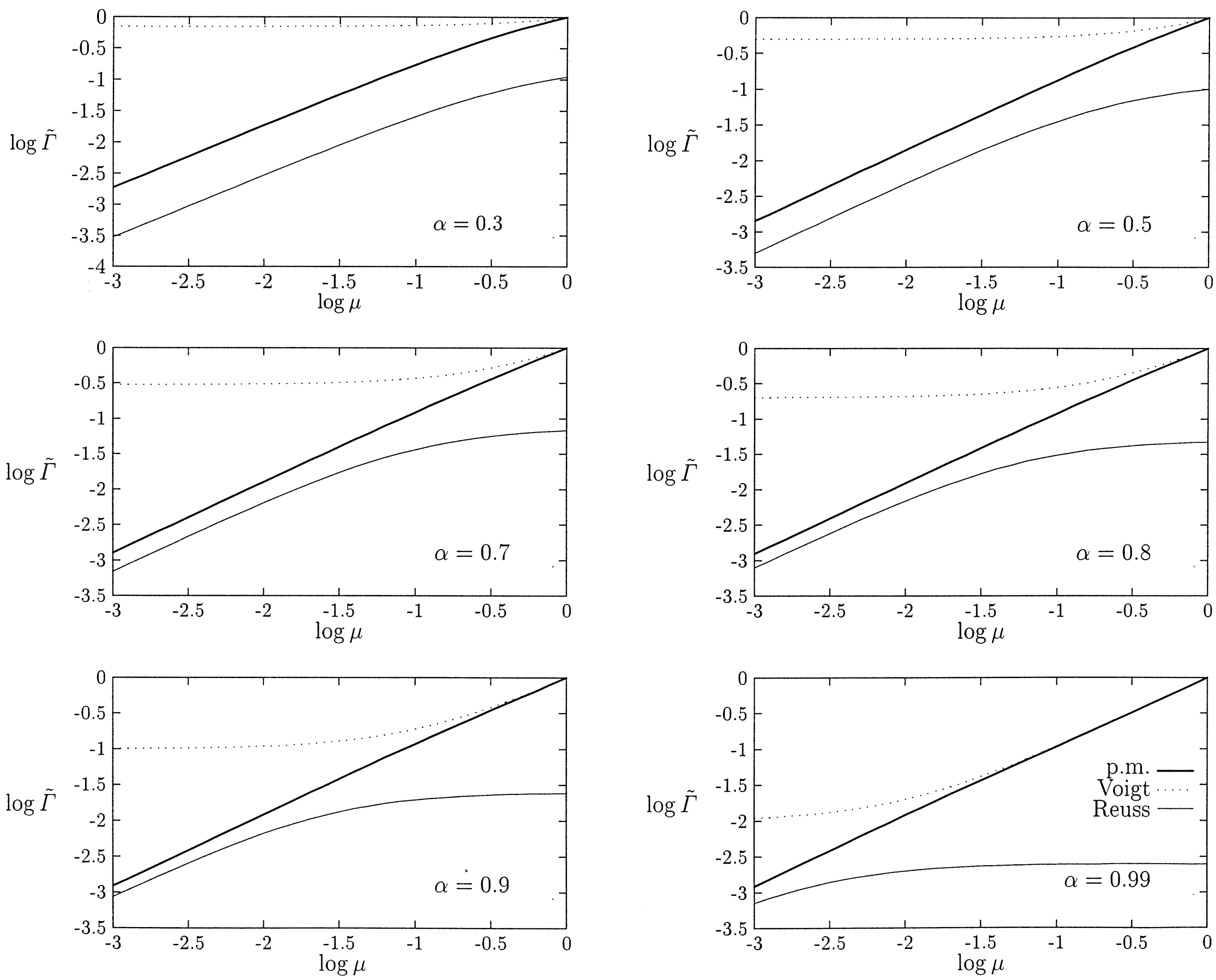

Fig. 4. Normalized transverse shear stiffnesses $\tilde{\Gamma}$ calculated by different models for a given ratio α (all stiffnesses are normalized by $\pi^2 G_{fh}/12$, p.m.—present model).

structure in Section 4.2. For different values of the shear moduli and the thicknesses, three transverse shear stiffnesses are calculated by

- Eqs. (35) and (36) (Γ),
- acting in parallel model

$$\Gamma_{\mathrm{P}} = \frac{\pi^2}{12}(G_f h_f + G_c h_c),$$

- acting in series model

$$\Gamma_{\mathrm{S}} = \frac{\pi^2}{12}\frac{G_f h_f G_c h_c}{G_f h_f + 4G_c h_c}.$$

Fig. 4 shows some calculations of normalized transverse shear stiffnesses for a given ratio α. From these calculations, it follows that the acting in parallel model is the upper bound and the acting in series model is the lower bound for the transverse shear stiffness. In the case of typical sandwich values, the effective transverse shear stiffness can be computed approximately by the Reuss model, otherwise for layered structures not far from homogeneous plates, this stiffness can be computed by the Voigt model.

5. Conclusions

The quality of structural mechanics analysis of thin sandwich and laminated plates by an equivalent layer theory is more influenced by the transverse shear stiffness than in the case of thin plates which are homogeneous in thickness direction. In the present paper, the question of unified estimation of the transverse shear stiffness for sandwiches and laminates is discussed. It is shown that the traditional models of Voigt and Reuss yield inaccurate results in some situations. The Voigt model, which can be recommended for any membrane, coupled and plate stiffnesses, only works in the case of layered structures with equal thicknesses and approximately identical elastic properties in each layer. For typical sandwiches, the transverse shear stiffness should be computed by the Reuss model. The other effective stiffnesses follow again from the Voigt model. The proposed method of calculating the transverse stiffnesses works in both situations. In addition, following the discussion of this work, the effective stiffnesses of viscoelastic layered system can be obtained by similar formulae. Further investigations should be directed to the question of how the proposed stiffnesses influence the accuracy of the structural analysis of laminates. In addition, the present approach should be compared with shear correction free results (e.g. Reddy and Robbins, 1994; Touratier, 1991).

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References

Altenbach, H., 1987. Definition of elastic moduli for plates made from thickness-uneven anisotropic material. Mech. Solids 22 (1), 135–141.

Altenbach, H., Altenbach, J., Rikards, R., 1996. Einführung in die Mechanik der Laminat- und Sandwichtragwerke. Deutscher Verlag für Grundstoffindustrie, Stuttgart.

Altenbach, H., Zhilin, P., 1988. Obshchaya teoriya uprugikh prostykh obolochek (in Russian). Uspekhi Mekhaniki 11 (4), 107–148.

Bert, C., 1973. Simplified analysis of static shear factors for beams of nonhomogeneous cross section. J. Composite Materials 7, 525–529.

Bollé, L., 1947a. Contribution au problème linéaire de flexin d'une plaque élastique. Bull. Techn. Suisse Romande 73 (21), 281–285.

Bollé, L., 1947b. Contribution au problème linéaire de flexin d'une plaque élastique. Bull. Techn. Suisse Romande 73 (22), 293–298.

Chawla, K., 1987. Composite Materials. Springer, Berlin.

Chow, T., 1971. On the propagation of flexural waves in an orthotropic laminated plate and its response to an impulsive load. J. Composite Materials 5, 306–319.

Christensen, R., 1979. Mechanics of Composite Materials. Wiley, New York.

Gould, P., 1988. Analysis of Shells and Plates. Springer, New York.

Hencky, H., 1947. Über die Berücksichtigung der Schubverzerrung in ebenen Platten. Ingenieur-Archiv 16, 72–76.

Hill, R., 1964. Theory of mechanical properties of fibre-strengthened materials: 1. elastic behaviour. J. Mech. Phys. Solids 12 (4), 199–218.

Kirchhoff, G., 1850. Über das Gleichgewicht und die Bewegung einer elastischen Scheibe. J. Reine Angew. Math. 40, 51–88.

Knight Jr., N., Qi, Y., 1997. Restatement of first-order shear-deformation theory for laminated plates. Int. J. Solids Structures 34 (4), 481–492.

Lai, W., Rubin, D., Krempl, E., 1993. Introduction to Continuum Mechanics. Pergamon Press, Oxford.

Love, A., 1944. A Treatise on the Mathematical Theory of Elasticity. Dover, New York.

Lurie, A., 1990. Nonlinear Theory of Elasticity. North Holland, Amsterdam.

Mindlin, R., 1951. Influence of rotatory inertia and shear on flexural motions of isotropic, elastic plates. Trans. ASME. J. Appl. Mech. 18 (2), 31–38.

Qi, Y., Knight Jr., N., 1996. A refined first-order shear-deformation theory and its justification by plane-strain bending problem of laminated plates. Int. J. Solids Structures 33 (1), 49–64.

Reddy, J., Robbins Jr., D., 1994. Theories and computational models for composite laminates. Appl. Mech. Rev. 47 (6), 21–35 (Part 1).

Reissner, E., 1944. On the theory of bending of elastic plates. J. Math. Phys. 23, 184–191.

Reissner, E., 1945. The effect of transverse shear deformation on the bending of elastic plates. J. Appl. Mech. 12 (11), A69–A77.

Reissner, E., 1947. On the bending of elastic plates. Quart. Appl. Math. 5, 55–68.

Rikards, R., Chate K, A., Kenzer, M., 1990. Varianty usredneniya sdvigovoi zhestkosti sloistykh konstrukcii pri raschete sobstvennykh chastot po modeli timoshenko (in Russian). Voprosy Dinamiki i Prochnosti 52, 176–193.

Timoshenko, S., 1921. On the correction for shear of the differential equation for transverse vibrations of prismatic bars. Phil. Mag. 41 (245), 744–746 (Ser. 6).

Touratier, M., 1991. An efficient standard plate theory. Int. J. Engng. Sci. 29 (8), 901–916.

Vlachoutsis, S., 1992. Shear correction factors for plates and shells. Int. J. Numer. Meth. Engng. 33, 1537–1552.

Whitney, J., 1973. Shear correction factors for orthotropic laminates under static load. Trans. ASME. J. Appl. Mech. 37 (1), 302–304.

Zhilin, P., 1976. Mechanics of deformable directed surfaces. Int. J. Solids Struct. 12, 635–648.